Periodic solution of a Lotka–Volterra predator–prey model with dispersion and time delays

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Abstract

A periodic Lotka–Volterra predator–prey model with dispersion and time delays is investigated. By using Gaines and Mawhin’s continuation theorem of coincidence degree theory and by means of a suitable Lyapunov functional, a set of easily verifiable sufficient conditions are derived for the existence, uniqueness and global stability of positive periodic solutions of the system. Sufficient conditions are also established for the uniform persistence of the system. Numerical simulations are presented to illustrate our main results.

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1. Introduction

Interest has been growing in the study of mathematical models of populations dispersing among patches in a heterogeneous environment ([1–25] and references cited therein). Many of the existing models deal with a single population dispersing among patches. Some of them deal with competition and predator–prey interactions in patchy environments. Most of the previous works deal with autonomous population systems. The analysis of these models has been centered around the coexistence of populations and the stability of

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equilibria. These works indicate that a diffusion process in an ecological system is often considered to have a stabilizing influence on the system [11,19], but is also probably destabilizing the system [13,14].

However, any biological or environmental parameters are naturally subject to fluctuation in time. The effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Thus, the assumptions of periodicity of the parameters are a way of incorporating the periodicity of the environment (such as seasonal effects of weather, food supplies, mating habits and so forth).

On the other hand, time delays of one type or another have been incorporated into biological models by many researchers, we refer to the monographs of Cushing [26], Gopalsamy [27], Kuang [28], and MacDonald [29] for general delayed biological systems and to Beretta and Kuang [30], Gopalsamy [31,32], Hastings [33], May [34], Ruan [35], Wang and Ma [36] and the references cited therein for studies on delayed biological systems. In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable and cause the population to fluctuate. Time delay due to gestation is a common example, because generally the consumption of prey by the predator throughout its past history governs the present birth rate of the predator. Therefore, more realistic models of population interactions should take into account the periodicity of the changing environment, dispersal and the effect of time delays.

The objective of this paper is to study the combined effects of dispersion, periodicity of the environment and delays on the dynamics of predator–prey systems. To do so we discuss the following periodic predator–prey system with dispersion and time delays

\[
\begin{align*}
x_1(t) &= x_1(t)[r_1(t) - a_{11}(t)x_1(t) - a_{13}(t)y(t)] + D_1(t)(x_2(t) - x_1(t)), \\
x_2(t) &= x_2(t)[r_2(t) - a_{22}(t)x_2(t)] + D_2(t)(x_1(t) - x_2(t)), \\
y(t) &= y(t)[-r_3(t) + a_{33}(t)x_1(t - \tau_1) - a_{33}(t)y(t - \tau_2)],
\end{align*}
\]

with initial conditions

\[
\begin{align*}
x_i(\theta) &= \phi_i(\theta), \quad y(\theta) = \psi(\theta), \quad \theta \in [-\tau, 0], \\
\phi_i(0) > 0, \quad \psi(0) > 0, \quad \phi_i, \psi \in C([-\tau, 0), R_+), \quad i = 1, 2,
\end{align*}
\]

where \(x_1(t)\) and \(y(t)\) denote the densities of species \(x\) and \(y\) in patch 1, respectively, and \(x_2(t)\) denotes the density of species \(x\) in patch 2. Predator species is confined to patch 1 while the prey species can disperse between two patches. \(r_i(t)\) is the intrinsic growth rate of the prey at patch \(i, i = 1, 2\); \(a_{ii}(t)\) \((i = 1, 2)\) are the density-dependent coefficients of the prey at patch \(i\); \(a_{13}(t)\) is the capturing rate of the predator; \(a_{31}(t)/a_{13}(t)\) is the conversion rate of nutrients into
the production rate of the predator, \( r_3(t) \) is the death rate of the predator; \( D_i(t) \) is dispersion rate of prey species \( x_i \), \( i = 1, 2 \). \( \tau = \max\{\tau_1, \tau_2\} \). \( \tau_1 \) is the delay due to gestation, that is, mature adult predators can only contribute to the production of predator biomass. In addition, we have included the term \(-a_{33}(t)y(t - \tau_2)\) in the dynamics of predator \( y \) to incorporate the negative feedback of predator crowding.

In this paper, for system (1.1) we always assume that:

(H1) \( r_i(t), a_{ij}(t) \ (i, j = 1, 2, 3), D_1(t) \) and \( D_2(t) \) are continuously positive periodic functions with period \( \omega \); \( \tau_1 \) and \( \tau_2 \) are non-negative constants.

It is well known that by the fundamental theory of functional differential equations [37], system (1.1) has a unique solution \( z(t) = (x_1(t), x_2(t), y(t)) \) satisfying initial conditions (1.2). By suitable modifications of standard technique in [1,10], it is easy to show that all solutions of system (1.1) corresponding to initial conditions (1.2) are defined on \([0, +\infty)\) and remain positive for all \( t \geq 0 \). In this paper, the solution of system (1.1) satisfying initial conditions (1.2) is said to be positive.

The organization of this paper is as follows. In the next section, by using Gaines and Mawhin’s continuation theorem of coincidence degree theory, we discuss the existence of positive \( \omega \)-periodic solutions of system (1.1) and (1.2). In Section 3, results on the boundedness of solutions and uniform persistence of the system are presented. In Section 4, by means of a suitable Lyapunov functional, a set of easily verifiable sufficient conditions are derived for the uniqueness and global stability of the positive periodic solution of the system. Finally, numerical simulations are presented to show the feasibility of the conditions in our main results.

2. Existence of periodic solutions

In order to obtain the existence of positive periodic solutions of (1.1), for convenience, we shall summarize in the following a few concepts and results from [38] that will be basic for this section.

Let \( X, Y \) be real Banach spaces, let \( L : \text{Dom} L \subset X \to Y \) be a linear mapping, and \( N : X \to Y \) a continuous mapping. The mapping \( L \) is called a Fredholm mapping of index zero if \( \dim \text{Ker} L = \text{codim} \text{Im} L < +\infty \) and \( \text{Im} L \) is closed in \( Y \). If \( L \) is a Fredholm mapping of index zero and there exist continuous projectors \( P : X \to X \), and \( Q : Y \to Y \) such that \( \text{Im} P = \text{Ker} L, \ \text{Ker} Q = \text{Im} L = \text{Im}(I - Q) \), then the restriction \( L_P \) of \( L \) to \( \text{Dom} L \cap \text{Ker} P : (I - P)X \to \text{Im} L \) is invertible. Denote the inverse of \( L_P \) by \( K_P \). If \( \Omega \) is an open bounded subset of \( X \), the mapping \( N \) will be called \( L \)-compact on \( \overline{\Omega} \) if \( QN(\overline{\Omega}) \) is bounded and
K_{\rho}(I - Q)N : \overline{\Omega} \to X is compact. Since \text{Im} Q is isomorphic to \text{Ker} L, there exists isomorphisms \text{J} : \text{Im} Q \to \text{Ker} L.

**Lemma 2.1.** Let \( \Omega \subset X \) be an open bounded set. Let \( L \) be a Fredholm mapping of index zero and \( N \) be \( L \)-compact on \( \overline{\Omega} \). Assume

(a) for each \( \lambda \in (0, 1), x \in \partial \Omega \cap \text{Dom} L, Lx \neq \lambda Nx; \\
(b) for each \( x \in \partial \Omega \cap \text{Ker} L, QNx \neq 0; \\
(c) \text{deg}\{JQN, \Omega \cap \text{Ker} L, 0\} \neq 0.

Then \( Lx = Nx \) has at least one solution in \( \overline{\Omega} \cap \text{Dom} L \).

In what follows we shall use the notations:

\[ \tilde{f} = \frac{1}{\omega} \int_{0}^{\omega} f(t) \, dt, \quad f^L = \min_{t \in [0, \omega]} f(t), \quad f^M = \max_{[0, \omega]} f(t), \]

where \( f \) is a continuous \( \omega \)-periodic function.

We are now in a position to state our result on the existence of a periodic solution of system (1.1).

**Theorem 2.1.** In addition to (H1), assume further that the following hold:

(H2) \( a_3 \left( r_1 - D_1 \right) - r_3 \alpha_1 > 0, \)

(H3) \( (r_2 - D_2) > 0. \)

Then system (1.1) has at least one positive \( \omega \)-periodic solution.

**Proof.** Let

\[ u_1(t) = \ln[x_1(t)], \quad u_2(t) = \ln[x_2(t)], \quad u_3(t) = \ln[y(t)], \quad (2.1) \]

On substituting (2.1) into (1.1), we rewrite (1.1) in the form

\[ \frac{du_1(t)}{dt} = r_1(t) - D_1(t) - a_{11}(t)e^{u_1(t)} - a_{13}(t)e^{u_3(t)} + D_1(t)e^{u_2(t)} - u_1(t), \]

\[ \frac{du_2(t)}{dt} = r_2(t) - D_2(t) - a_{22}(t)e^{u_2(t)} + D_2(t)e^{u_1(t)} - u_2(t), \]

\[ \frac{du_3(t)}{dt} = -r_3(t) + a_{31}(t)e^{u_1(t-t_1)} - a_{33}(t)e^{u_3(t-t_2)}. \]

It is easy to see that if system (2.2) has one \( \omega \)-periodic solution \((u_1^*(t), u_2^*(t), u_3^*(t))^T\), then \( x^*(t) = (x_1^*(t), x_2^*(t), y^*(t))^T = (\exp[u_1^*(t)], \exp[u_2^*(t)], \exp[u_3^*(t)])^T \) is a positive \( \omega \)-periodic solution of system (1.1). Therefore, to complete the proof, it suffices to show that system (2.2) has one \( \omega \)-periodic solution.
Take
\[ X = Y = \{(u_1(t), u_2(t), u_3(t))^T \in C(R, R^3) : u_i(t + \omega) = u_i(t), i = 1, 2, 3\} \]
and
\[ \|(u_1(t), u_2(t), u_3(t))^T\| = \sum_{i=1}^{3} \max_{t \in [0,\omega]} |u_i(t)|, \]
here \(| \cdot |\) denotes the Euclidean norm. Then \( X \) and \( Y \) are Banach spaces with the norm \( \| \cdot \| \). Set
\[ L : \text{Dom} L \cap X, \quad L(u_1(t), u_2(t), u_3(t))^T = \left( \frac{du_1(t)}{dt}, \frac{du_2(t)}{dt}, \frac{du_3(t)}{dt} \right)^T, \]
where \( \text{Dom} L = \{(u_1(t), u_2(t), u_3(t))^T \in C^1(R, R^3)\} \) and \( N : X \to X \),
\[ N \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} r_1(t) - D_1(t) - a_{11}(t)e^{u_1(t)} - a_{13}(t)e^{u_3(t)} + D_1(t)e^{u_2(t)} - u_1(t) \\ r_2(t) - D_2(t) - a_{22}(t)e^{u_2(t)} + D_2(t)e^{u_1(t)} - u_2(t) \\ -r_3(t) + a_{31}(t)e^{u_1(t)} - a_{33}(t)e^{u_3(t)} \end{bmatrix}. \]
Define two projectors \( P \) and \( Q \) as
\[ P \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = Q \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega} \int_0^\omega u_1(t) \, dt \\ \frac{1}{\omega} \int_0^\omega u_2(t) \, dt \\ \frac{1}{\omega} \int_0^\omega u_3(t) \, dt \end{bmatrix}, \quad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in X = Y. \]
It is not difficult to show that
\[ \text{Ker} L = \{x | x \in X, x = h, h \in R^3\}, \]
\[ \text{Im} L = \left\{y | y \in Y, \int_0^\omega y(t) \, dt = 0\right\} \text{ is closed in } Y, \]
and
\[ \text{dim Ker} L = \text{codim Im} L = 3, \]
and \( P \) and \( Q \) are continuous projectors such that
\[ \text{Im} P = \text{Ker} L, \quad \text{Ker} Q = \text{Im} L = \text{Im}(I - Q). \]
It follows that \( L \) is a Fredholm mapping of index zero. Furthermore, the inverse \( K_P \) of \( L_P \) has the form \( K_P : \text{Im} L \to \text{Dom} L \cap \text{Ker} P, \)
\[ K_P(y) = \int_0^t y(s) \, ds - \frac{1}{\omega} \int_0^\omega \int_0^t y(s) \, ds \, dt. \]
Then $QN : X \to Y$ and $K_P(I - Q)N : X \to X$ read

$$QN_x = \left[ \frac{1}{\omega} \int_0^\omega \left[ r_1(t) - D_1(t) - a_{11}(t)e^{u_1(t)} - a_{13}(t)e^{u_3(t)} + D_1(t)e^{u_2(t) - u_1(t)} \right] \, dt \right] - \frac{1}{\omega} \int_0^\omega \left[ r_2(t) - D_2(t) - a_{22}(t)e^{u_2(t)} + D_2(t)e^{u_1(t) - u_2(t)} \right] \, dt$$

$$\quad - \frac{1}{\omega} \int_0^\omega \left[ -r_3(t) + a_{31}(t)e^{u_1(t)} - a_{33}(t)e^{u_3(t)} \right] \, dt,$$

$$K_P(I - Q)N_x = \int_0^t N_x(s) \, ds - \frac{1}{\omega} \int_0^\omega \int_0^t N_x(s) \, ds \, dt$$

$$\quad - \left( \frac{t}{\omega} - \frac{1}{2} \right) \int_0^\omega N_x(s) \, ds.$$

Clearly, $QN$ and $K_P(I - Q)N$ are continuous. By using the Arzela–Ascoli theorem, it is not difficult to prove that $K_P(I - Q)N(\bar{Q})$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\bar{Q})$ is bounded. Therefore, $N$ is $L$-compact on $\bar{Q}$ with any open bounded set $\Omega \subset X$.

In order to apply Lemma 2.1, we need to search for an appropriate open, bounded subset $\Omega$.

Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0, 1)$, we have

$$\frac{du_1(t)}{dt} = \lambda \left[ r_1(t) - D_1(t) - a_{11}(t)e^{u_1(t)} - a_{13}(t)e^{u_3(t)} + D_1(t)e^{u_2(t) - u_1(t)} \right],$$

$$\frac{du_2(t)}{dt} = \lambda \left[ r_2(t) - D_2(t) - a_{22}(t)e^{u_2(t)} + D_2(t)e^{u_1(t) - u_2(t)} \right],$$

$$\frac{du_3(t)}{dt} = \lambda \left[ -r_3(t) + a_{31}(t)e^{u_1(t)} - a_{33}(t)e^{u_3(t)} \right]. \quad (2.3)$$

Suppose that $(u_1(t), u_2(t), u_3(t))^T \in X$ is a solution of (2.3) for a certain $\lambda \in (0, 1)$. Integrating (2.3) over the interval $[0, \omega]$ leads to

$$\int_0^\omega a_{11}(t)e^{u_1(t)} \, dt + \int_0^\omega a_{13}(t)e^{u_3(t)} \, dt = \int_0^\omega (r_1(t) - D_1(t)) \, dt + \int_0^\omega D_1(t)e^{u_2(t) - u_1(t)} \, dt, \quad (2.4)$$

$$\int_0^\omega a_{22}(t)e^{u_2(t)} \, dt = \int_0^\omega (r_2(t) - D_2(t)) \, dt + \int_0^\omega D_2(t)e^{u_1(t) - u_2(t)} \, dt, \quad (2.5)$$

$$\int_0^\omega a_{31}(t)e^{u_1(t)} \, dt = \int_0^\omega r_3(t) \, dt + \int_0^\omega a_{33}(t)e^{u_3(t)} \, dt. \quad (2.6)$$
It follows from (2.3)–(2.6) that
\[
\int_0^\omega |u_1'(t)|\,dt < \int_0^\omega [r_1(t) - D_1(t) + a_{11}(t)e^{u_1(t)} + a_{13}(t)e^{u_3(t)} + D_1(t)e^{u_2(t) - u_1(t)}]\,dt
\]
\[
+ 2 \int_0^\omega a_{11}(t)e^{u_1(t)}\,dt + 2 \int_0^\omega a_{13}(t)e^{u_3(t)}\,dt,
\]
\[
\int_0^\omega |u_2'(t)|\,dt < \int_0^\omega [r_2(t) - D_2(t) + a_{22}(t)e^{u_2(t)} + D_2(t)e^{u_3(t) - u_2(t)}]\,dt
\]
\[
= 2 \int_0^\omega a_{22}(t)e^{u_2(t)}\,dt,
\]
\[
\int_0^\omega |u_3'(t)|\,dt < \int_0^\omega [r_3(t) + a_{31}(t)e^{u_1(t - \tau_1)} + a_{33}(t)e^{u_1(t - \tau_2)}]\,dt
\]
\[
= 2 \int_0^\omega a_{31}(t)e^{u_1(t - \tau_1)}\,dt.
\] (2.7)

Multiplying the first equation of (2.3) by \(e^{u_1(t)}\) and integrating over \([0, \omega]\) gives
\[
\int_0^\omega a_{11}(t)e^{2u_1(t)}\,dt < \int_0^\omega (r_1(t) - D_1(t))e^{u_1(t)}\,dt + \int_0^\omega D_1(t)e^{u_2(t)}\,dt,
\]
which yields
\[
a_{11}^L \int_0^\omega e^{2u_1(t)}\,dt < (r_1 - D_1)^M \int_0^\omega e^{u_1(t)}\,dt + D_1^M \int_0^\omega e^{u_2(t)}\,dt.
\] (2.8)

Similarly, multiplying the second equation in (2.3) by \(e^{u_2(t)}\) and integrating over \([0, \omega]\) gives
\[
\int_0^\omega a_{22}(t)e^{2u_2(t)}\,dt < \int_0^\omega (r_2(t) - D_2(t))e^{u_2(t)}\,dt + \int_0^\omega D_2(t)e^{u_1(t)}\,dt,
\]
which implies
\[
a_{22}^L \int_0^\omega e^{2u_2(t)}\,dt < (r_2 - D_2)^M \int_0^\omega e^{u_2(t)}\,dt + D_2^M \int_0^\omega e^{u_1(t)}\,dt.
\] (2.9)

By using the inequalities
\[
\left( \int_0^\omega e^{u_i(t)}\,dt \right)^2 \leq \omega \int_0^\omega e^{2u_i(t)}\,dt, \quad i = 1, 2,
\]
it follows from (2.8) and (2.9) that
\[
a_{11}^L \left( \int_0^\omega e^{u_1(t)}\,dt \right)^2 < \omega (r_1 - D_1)^M \int_0^\omega e^{u_1(t)}\,dt + D_1^M \omega \int_0^\omega e^{u_2(t)}\,dt,
\] (2.10)
\[
a_{22}^L \left( \int_0^\omega e^{u_2(t)}\,dt \right)^2 < \omega (r_2 - D_2)^M \int_0^\omega e^{u_2(t)}\,dt + D_2^M \omega \int_0^\omega e^{u_1(t)}\,dt.
\] (2.11)
If $\int_0^\alpha e^{\mu_2(t)} \, dt \leq \int_0^\alpha e^{\mu_1(t)} \, dt$, then it follows from (2.10) that
\[
a_{11}^L \left( \int_0^\alpha e^{\mu_1(t)} \, dt \right)^2 < \alpha (r_1 - D_1)^M \int_0^\alpha e^{\mu_1(t)} \, dt + D_1^M \alpha \int_0^\alpha e^{\mu_1(t)} \, dt,\]
which leads to
\[
\int_0^\alpha e^{\mu_2(t)} \, dt \leq \int_0^\alpha e^{\mu_1(t)} \, dt < \frac{\alpha (r_1 - D_1)^M + \alpha D_1^M}{a_{11}^L}. \tag{2.12}
\]
If $\int_0^\alpha e^{\mu_1(t)} \, dt \leq \int_0^\alpha e^{\mu_2(t)} \, dt$, then it follows from (2.11) that
\[
a_{22}^L \left( \int_0^\alpha e^{\mu_2(t)} \, dt \right)^2 < \alpha (r_2 - D_2)^M \int_0^\alpha e^{\mu_2(t)} \, dt + D_2^M \alpha \int_0^\alpha e^{\mu_2(t)} \, dt,\]
which yields
\[
\int_0^\alpha e^{\mu_1(t)} \, dt \leq \int_0^\alpha e^{\mu_2(t)} \, dt < \frac{\alpha (r_2 - D_2)^M + \alpha D_2^M}{a_{22}^L}. \tag{2.13}
\]
Set
\[
A = \max \left\{ \frac{(r_1 - D_1)^M + D_1^M}{a_{11}^L}, \frac{(r_2 - D_2)^M + D_2^M}{a_{22}^L} \right\}. \tag{2.14}
\]
Then it follows from (2.12)–(2.14) that
\[
\int_0^\alpha e^{\mu_i(t)} \, dt < \alpha A, \quad i = 1, 2. \tag{2.15}
\]
Since $(u_1(t), u_2(t), u_3(t))^T \in X$, there exist $\xi_i, \eta_i \in [0, \alpha]$ such that
\[
u_i(\xi_i) = \min_{t \in [0, \alpha]} u_i(t), \quad u_i(\eta_i) = \max_{t \in [0, \alpha]} u_i(t), \quad i = 1, 2, 3. \tag{2.16}
\]
It follows from (2.15) and (2.16) that
\[
u_i(\xi_i) < \ln A, \quad i = 1, 2. \tag{2.17}
\]
Noticing that
\[
\int_0^\alpha e^{\nu_i(t-\tau_i)} \, dt = \int_0^\alpha e^{\nu_i(t)} \, dt, \quad \int_0^\alpha e^{\nu_3(t-\tau_2)} \, dt = \int_0^\alpha e^{\nu_3(t)} \, dt, \tag{2.18}
\]
we derive from (2.6) and (2.15) that
\[
\int_0^\alpha a_{33}(t)e^{\nu_3(t-\tau_2)} \, dt \leq \int_0^\alpha a_{31}(t)e^{\mu_1(t-\tau_1)} \, dt
\leq a_{31}^M \int_0^\alpha e^{\nu_1(t)} \, dt = a_{31}^M \int_0^\alpha e^{\mu_1(t)} \, dt < a_{31}^M \alpha A,
\]
which implies
\[
\int_0^\alpha e^{u_3(t)} \, dt \leq \frac{a_{31}^M \omega A}{d_{33}^M}
\]  
(2.19)

and

\[
u_3(\xi_3) \leq \ln \frac{a_{31}^M A}{d_{33}^M}.
\]  
(2.20)

It follows from (2.7), (2.15), (2.18) and (2.19) that
\[
\int_0^\alpha |u'_1(t)| \, dt < 2\alpha A \left( a_{11}^M + \frac{a_{13}^M a_{31}^M}{d_{33}^M} \right) = c_1,
\]
\[
\int_0^\alpha |u'_2(t)| \, dt < 2a_{22}^M \omega A,
\]  
(2.21)

\[
\int_0^\alpha |u'_3(t)| \, dt < 2a_{31}^M \omega A.
\]

Then from (2.17), (2.20) and (2.21) we obtain
\[
u_1(t) \leq u_1(\xi_1) + \int_0^\alpha |u'_1(t)| \, dt < \ln A + c_1,
\]  
(2.22)

\[
u_2(t) \leq u_2(\xi_2) + \int_0^\alpha |u'_2(t)| \, dt < \ln A + 2a_{22}^M \omega A,
\]  
(2.23)

\[
u_3(t) \leq u_3(\xi_3) + \int_0^\alpha |u'_3(t)| \, dt < \ln \frac{a_{31}^M A}{d_{33}^M} + 2a_{31}^M \omega A.
\]  
(2.24)

It follows from (2.6) that
\[
\int_0^\alpha a_{31}(t)e^{u_1(t-\tau_1)} \, dt > \int_0^\alpha r_3(t) \, dt,
\]
which implies
\[
u_1(\eta_1) > \ln \frac{\tau_3}{a_{31}^M}.
\]  
(2.25)

This, together with (2.21), leads to
\[
u_1(t) \geq \nu_1(\eta_1) - \int_0^\alpha |u'_1(t)| \, dt > \ln \frac{\tau_3}{a_{31}^M} - c_1.
\]  
(2.26)

It follows from (2.22) and (2.26) that
\[
\max_{t \in [0, \alpha]} |u_1(t)| < \max \left\{ \left| \ln A \right| + c_1, \left| \ln \frac{\tau_3}{a_{31}^M} \right| + c_1 \right\} := R_1.
\]  
(2.27)
From (2.5) we can easily derive
\[ u_2(\eta_2) \geq \ln \frac{(r_2 - D_2)}{a_{22}^2}. \] (2.28)

It follows from (2.21) and (2.28) that
\[ u_2(t) \geq u_2(\eta_2) - \int_0^t |u_2'(s)| \, ds \geq \ln \frac{(r_2 - D_2)}{a_{22}^2} - 2a_{22}^2 \omega_0. \] (2.29)

This, together with (2.23), leads to
\[ \max_{t \in [0, \infty]} |u_2(t)| < \max \left\{ |\ln A| + 2a_{22}^2 \omega_0, \left| \frac{(r_2 - D_2)}{a_{22}^2} \right| + 2a_{22}^2 \omega_0 \right\} := R_2. \] (2.30)

Noticing that
\[ (r_1 - D_1) \omega \leq a_{11}^M \int_0^\omega e^{u_1(s)} \, ds + a_{13}^M \int_0^\omega e^{u_3(s)} \, ds, \] (2.31)
we derive from (2.6) and (2.18) that
\[ a_{33}^M \int_0^\omega e^{u_3(s)} \, ds = a_{33}^M \int_0^\omega e^{u_3(t-\tau_2)} \, dt \]
\[ \geq a_{31}^L \int_0^\omega e^{u_1(t-\tau_1)} \, dt - \tau_3 \omega \]
\[ = a_{31}^L \int_0^\omega e^{u_1(t)} \, dt - \tau_3 \omega \]
\[ \geq a_{31}^L \frac{(r_1 - D_1) \omega - a_{15}^M \int_0^\omega e^{u_3(s)} \, ds}{a_{11}^M} - \tau_3 \omega, \] (2.32)
which implies
\[ \int_0^\omega e^{u_3(t)} \, dt \geq \omega a_{31}^L (r_1 - D_1) - \tau_3 a_{11}^M \]
\[ a_{11}^M a_{33}^M + a_{13}^M a_{51}^M \]
and
\[ u_3(\eta_3) \geq \ln \frac{a_{31}^L (r_1 - D_1) - \tau_3 a_{11}^M}{a_{11}^M a_{33}^M + a_{13}^M a_{51}^M} \equiv d_3. \] (2.33)

It follows from (2.21) and (2.33) that
\[ u_3(t) \geq u_3(\eta_3) - \int_0^t |u_3'(s)| \, ds > d_3 - 2a_{31}^M \omega_0. \] (2.34)
This, together with (2.24), leads to
\[
\max_{t \in [0,\omega]} |u_1(t)| < \max \left\{ \left| \ln \frac{a_{31}M}{a_{33}} \right| + 2a_{31}M \omega A, |d_3| + 2a_{31}M \omega A \right\} := R_3. \tag{2.35}
\]

Clearly, \( R_1, R_2 \) and \( R_3 \) in (2.27), (2.30) and (2.35) are independent of \( \lambda \). Denote \( M = R_1 + R_2 + R_3 + R_0 \), where \( R_0 \) is taken sufficiently large such that each solution \( (x^*, \beta^*, \gamma^*)^T \) of the system of algebraic equations
\[
\begin{align*}
(r_1 - D_1) - \tilde{a}_{11} e^x - \tilde{a}_{13} e^y + D_1 e^{\beta - x} &= 0, \\
(r_2 - D_2) - \tilde{a}_{22} e^\beta + D_2 e^{x - \beta} &= 0, \\
- \tilde{r}_3 + \tilde{a}_{31} e^x - \tilde{a}_{33} e^y &= 0,
\end{align*}
\tag{2.36}
\]
satisfies \( \|(x^*, \beta^*, \gamma^*)^T\| = \|x^*\| + |\beta^*| + |\gamma^*| < M \) (if system (2.36) has at least one solution) and
\[
\max \left\{ \left| \ln A_1 \right|, \left| \ln \frac{\tilde{r}_3}{a_{31}} \right| \right\} + \max \left\{ \left| \ln A_1 \right|, \left| \ln \frac{r_2 - D_2}{a_{22}} \right| \right\} + \max \left\{ \left| \ln \frac{a_{31}A_1}{a_{33}} \right|, \left| \ln \frac{a_{31}(r_1 - D_1) - \tilde{r}_3 a_{11}}{a_{33}} \right| \right\} < M, \tag{2.37}
\]

where
\[
A_1 = \max \left\{ \frac{(r_1 - D_1) + D_1}{a_{11}}, \frac{(r_2 - D_2) + D_2}{a_{22}} \right\}. \tag{2.38}
\]

We now take \( \Omega = \{(u_1(t), u_2(t), u_3(t))^T \in X : \|(u_1, u_2, u_3)^T\| < M\} \). This satisfies the condition (a) in Lemma 2.1. When \((u_1(t), u_2(t), u_3(t))^T \in \partial \Omega \cap \text{Ker} L = \partial \Omega \cap R^3, (u_1, u_2, u_3)^T\) is a constant vector in \( R^3 \) with \(|u_1| + |u_2| + |u_3| = M \). If system (2.36) has at least one solution, then we have
\[
QN \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} (r_1 - D_1) - \tilde{a}_{11} e^{u_1} - \tilde{a}_{13} e^{u_3} + D_1 e^{u_2 - u_1} \\ (r_2 - D_2) - \tilde{a}_{22} e^{u_2} + D_2 e^{u_1 - u_2} \\ - \tilde{r}_3 + \tilde{a}_{31} e^{u_1} - \tilde{a}_{33} e^{u_3} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

If system (2.36) does not have a solution, we can directly derive
\[
QN \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

This proves that condition (b) in Lemma 2.1 is satisfied.
In order to prove that condition (c) in Lemma 2.1 holds, we define $\phi : \text{Dom} X \times [0, 1] \to X$ by

$$
\phi(u_1, u_2, u_3, \mu) = \begin{bmatrix}
(r_1 - D_1) - \overline{a_{11}} e^{u_1} - \overline{a_{13}} e^{u_3} \\
(r_2 - D_2) - \overline{a_{22}} e^{u_2} \\
- \overline{r_3} + \overline{a_{31}} e^{u_1} - \overline{a_{33}} e^{u_3}
\end{bmatrix} + \mu \begin{bmatrix}
D_1 e^{u_2 - u_1} \\
D_2 e^{u_1 - u_2} \\
0
\end{bmatrix},
$$

where $\mu \in [0, 1]$ is a parameter. When $(u_1(t), u_2(t), u_3(t))^T \in \partial \Omega \cap \text{Ker} L = \partial \Omega \cap R^3$, $(u_1, u_2, u_3)^T$ is a constant vector in $R^3$ with $|u_1| + |u_2| + |u_3| = M$. We will show that when $(u_1, u_2, u_3)^T \in \partial \Omega \cap \text{Ker} L$, $\phi(u_1, u_2, u_3, \mu) \neq 0$. Assume the conclusion is not true, i.e., there is a constant vector $(u_1, u_2, u_3)^T$ with $|u_1| + |u_2| + |u_3| = M$ satisfying $\phi(u_1, u_2, u_3, \mu) = 0$, that is,

$$
\begin{align*}
(r_1 - D_1) - \overline{a_{11}} e^{u_1} - \overline{a_{13}} e^{u_3} + \mu D_1 e^{u_2 - u_1} &= 0, \\
(r_2 - D_2) - \overline{a_{22}} e^{u_2} + \mu D_2 e^{u_1 - u_2} &= 0, \\
- \overline{r_3} + \overline{a_{31}} e^{u_1} - \overline{a_{33}} e^{u_3} &= 0.
\end{align*}
$$

Similar arguments in (2.27), (2.30) and (2.35) show that

\[
|u_1| < \max \left\{ \left| \ln A_1 \right|, \left| \frac{\overline{r_3}}{\overline{a_{31}}} \right| \right\},
\]

\[
|u_2| < \max \left\{ \left| \ln A_1 \right|, \left| \frac{(r_2 - D_2)}{\overline{a_{22}}} \right| \right\},
\]

\[
|u_3| < \max \left\{ \left| \ln \frac{\overline{a_{31}} A_1}{\overline{a_{33}}} \right|, \left| \ln \frac{\overline{a_{31}} (r_1 - D_1) - \overline{r_3} a_{11}}{\overline{a_{13}} A_1 + \overline{a_{11}} \overline{a_{33}}} \right| \right\},
\]

where $A_1$ is defined by (2.38). Thus, it follows from (2.37) that

\[
|u_1| + |u_2| + |u_3| < M,
\]

which leads to a contradiction. Using the property of topological degree and taking $J = I : \text{Im} Q \to \text{Ker} L$, $(u_1, u_2, u_3)^T \to (u_1, u_2, u_3)^T$, we have

\[
\text{deg}(JQN(u_1, u_2, u_3)^T, \partial \Omega \cap \text{Ker} L, (0, 0, 0)^T)
\]

\[
= \text{deg}(\phi(u_1, u_2, u_3, 1), \partial \Omega \cap \text{Ker} L, (0, 0, 0)^T)
\]

\[
= \text{deg}(\phi(u_1, u_2, u_3, 0), \partial \Omega \cap \text{Ker} L, (0, 0, 0)^T)
\]

\[
= \text{deg}(\overline{(r_1 - D_1) - a_{11} e^{u_1} - a_{13} e^{u_3}}, \overline{(r_2 - D_2)} - \overline{a_{22}} e^{u_2}, - \overline{r_3} + \overline{a_{31}} e^{u_1} - \overline{a_{33}} e^{u_3})^T, \partial \Omega \cap \text{Ker} L, (0, 0, 0)^T).
\]
By (H2) and (H3), we see that the following system of algebraic equation
\[
\begin{align*}
(r_1 - D_1) - \alpha_{11}^u e^{u_1} - \alpha_{13}^u e^{u_3} &= 0, \\
(r_2 - D_2) - \alpha_{22}^u e^{u_2} &= 0, \\
-r_3 + \alpha_{31}^u e^{u_1} - \alpha_{33}^u e^{u_3} &= 0,
\end{align*}
\]
has a unique solution \((u_1^*, u_2^*, u_3^*)\) which satisfies
\[
\begin{align*}
u_1^* &= \ln \frac{\alpha_{33}(r_1 - D_1) + r_3}{\alpha_{11} \alpha_{33} + \alpha_{13} \alpha_{31}}, \\
u_2^* &= \ln \frac{(r_2 - D_2)}{\alpha_{22}}, \\
u_3^* &= \ln \frac{\alpha_{31}(r_1 - D_1) - \alpha_{11} r_3}{\alpha_{11} \alpha_{33} + \alpha_{13} \alpha_{31}}.
\end{align*}
\]
Thus, a standard and direct calculation shows that
\[
\text{deg}(JQN(u_1, u_2, u_3)^T, \Omega \cap \text{Ker} L, (0, 0, 0)^T) = -1.
\]
By now we have proved that \(\Omega\) satisfies all the requirements in Lemma 2.1. Hence, (2.2) has at least one \(\omega\)-periodic solution. Accordingly, system (1.1) has at least one positive \(\omega\)-periodic solution. The proof is complete. \(\square\)

3. Uniform persistence

In this section, we discuss the uniform persistence of system (1.1).

**Definition 3.1.** System (1.1) is said to be uniform persistent if there exists a compact region \(D \subset \text{Int} R^3_+\) such that every solution \(z(t)\) of (1.1) with initial conditions (1.2) eventually enters and remains in the region \(D\).

**Lemma 3.1.** Let \(z(t) = (x_1(t), x_2(t), y(t))\) denote any positive solution of system (1.1) with initial conditions (1.2). Then there exists a \(T_2 > 0\) such that
\[
0 < x_i(t) \leq M_i \quad (i = 1, 2); \quad 0 < y(t) \leq M_3 \quad \text{for} \ t \geq T_2,
\]
where
\[
M_1 = M_2 > M_1^*, \quad M_1^* = \max \left\{ \frac{r_1^M}{\alpha_{11}^M}, \frac{r_2^M}{\alpha_{22}^M} \right\}, \quad M_3 = \frac{a_{31}^M M_1}{\alpha_{33}^M} e^{a_{31}^M t_2}.
\]

**Proof.** We define
\[
V(t) = \max \{x_1(t), x_2(t)\}.
\]
Calculating the upper-right derivative of $V(t)$ along the positive solution of system (1.1), we have the following.

(P1) If $x_1(t) > x_2(t)$ or $x_1(t) = x_2(t)$, then
\[
D^+ V(t) = \dot{x}_1(t)
= x_1(t)[r_1(t) - a_{11}(t)x_1(t) - a_{13}(t)x_3(t)] + D_1(t)(x_2(t) - x_1(t))
\leq x_1(t)[r^M_1 - a^L_{11}x_1(t)];
\]

(P2) If $x_1(t) < x_2(t)$ or $x_1(t) = x_2(t)$, then
\[
D^+ V(t) = \dot{x}_2(t)
= x_2(t)[r_2(t) - a_{22}(t)x_2(t)] + D_2(t)(x_1(t) - x_2(t))
\leq x_2(t)[r^M_2 - a^L_{22}x_2(t)].
\]

It follows from (P1) and (P2) that
\[
D^+ V(t) \leq x_i(t)[r^M_i - a^L_i x_i(t)] \quad (i = 1 \text{ or } 2). \tag{3.3}
\]

By (3.3) we can derive

(A) If $\max\{x_1(0), x_2(0)\} \leq M_1$, then $\max\{x_1(t), x_2(t)\} \leq M_1$, $t \geq 0$.

(B) If $\max\{x_1(0), x_2(0)\} > M_1$. Let $-\alpha = \max_{i=1,2} \{M_1(r^M_i - a^L_i M_1)\}$ ($\alpha > 0$). We consider the following three possibilities:

(a) $V(0) = x_1(0) > M_1(x_1(0) > x_2(0))$;
(b) $V(0) = x_2(0) > M_1(x_1(0) < x_2(0))$;
(c) $V(0) = x_1(0) = x_2(0) > M_1$.

If (a) holds, then there exists $\varepsilon > 0$ such that if $t \in [0, \varepsilon)$, then $V(t) = x_1(t) > M_1$, and we have
\[
D^+ V(x_1(t), x_2(t)) = \dot{x}_1(t) < -\alpha < 0.
\]

If (b) holds, then there exists $\varepsilon > 0$ such that if $t \in [0, \varepsilon)$, $V(t) = x_2(t) > M_1$, and also we have
\[
D^+ V(x_1(t), x_2(t)) = \dot{x}_2(t) < -\alpha < 0.
\]

If (c) holds, a similar argument in (a) and (b) shows that there exists $\varepsilon > 0$ such that if $t \in [0, \varepsilon)$, then
\[
D^+ V(x_1(t), x_2(t)) = \dot{x}_i(t) < -\alpha < 0 \quad (i = 1 \text{ or } 2).
\]

From what has been discussed above, we can conclude that if $V(0) > M_1$, then $V(t)$ is strictly monotone decreasing with speed at least $\alpha$. Therefore there exists a $T_1 > 0$ such that if $t \geq T_1$, then
\[ V(t) = \max\{x_1(t), x_2(t)\} \leq M_1. \]  

(3.4)

In addition, from the third equation of system (1.1) and (3.4), we derive that for \( t > T_1 + \tau_1 \),

\[ y(t) \leq y(t)[a_{31}^M M_1 - a_{33}^L y(t - \tau_2)]. \]

A similar argument in the proof of Lemma 2.1 in [36] shows that there exists a \( T_2 \geq T_1 + \tau_1 \) such that

\[ y(t) \leq \frac{a_{31}^M M_1}{a_{33}^L} e^{a_{31}^M \tau_2} \defeq M_3 \quad \text{for} \ t \geq T_2. \]

This completes the proof. \( \qed \)

**Theorem 3.1.** Let (H1) hold. Assume further that

\[(H4)\quad a_{31}^L \min \left\{ \frac{r_1^L - a_{31}^M M_3}{a_{11}^M}, \frac{r_2^L}{a_{22}^M} \right\} > r_3^M.\]

Then system (1.1) is uniformly persistent, i.e., there exist positive constants \( T > T_2 \) and \( m_i (i = 1, 2, 3) \) such that

\[ m_i < x_i(t) < M_i \ (i = 1, 2), \quad m_3 < y(t) < M_3 \quad \text{for} \ t \geq T, \]

(3.5)

in which

\[ m_i = \frac{1}{2} m_i^*, \quad i = 1, 2, 3; \]

\[ m_i^* = m_2^* = \min \left\{ \frac{r_1^L - a_{31}^M M_3}{a_{11}^M}, \frac{r_2^L}{a_{22}^M} \right\}, \quad m_3^* = \frac{a_{31}^L M_1 - r_3^M}{a_{33}^M} \]

(3.6)

and \( M_i, M_2 \) and \( M_3 \) are defined in (3.2).

The proof is standard and similar to that of Theorem 3.2 of [25], we therefore omit it here.

### 4. Uniqueness and global stability of periodic solutions

In this section, we formulate the uniqueness and global stability of the \( \omega \)-periodic solution \( x^*(t) \) in Theorem 2.1. It is immediate that if \( x^*(t) \) is globally asymptotically stable then \( x^*(t) \) is in fact unique.

**Theorem 4.1.** In addition to (H1)–(H4), assume further that

\[(H5)\quad \liminf_{t \to \infty} A_i(t) > 0,\]

(H4)
where
\[ A_1(t) = a_{11}(t) - a_{31}(t + \tau_1) - \frac{D^M_2}{m_2} - a_{31}(t + \tau_1)M_3 \int_{t+\tau_1}^{t+\tau_1+\tau_2} a_{33}(s) \, ds; \]
\[ A_2(t) = a_{22}(t) - \frac{D^M_1}{m_1}; \]
\[ A_3(t) = a_{33}(t) - a_{13}(t) - (r_3(t) + a_{31}(t)M_1 + a_{33}(t)M_3) \int_t^{t+\tau_2} a_{33}(s) \, ds \]
\[ - a_{33}(t + \tau_2)M_3 \int_{t+\tau_2}^{t+2\tau_2} a_{33}(s) \, ds. \]

Then system (1.1) and (1.2) has a unique positive \( \omega \)-periodic solution \( x^*(t) = (x_1^*(t), x_2^*(t), y^*(t))^T \) which is globally asymptotically stable.

**Proof.** Due to the conclusion of Theorem 2.1, we only need to show the global asymptotic stability of the positive periodic solution of (1.1) and (1.2). Let \((x_1^*(t), x_2^*(t), y^*(t))^T\) be a positive \( \omega \)-periodic solution of system (1.1), and \((y_1(t), y_2(t), y_3(t))^T\) be any positive solution of system (1.1). It follows from Theorem 3.1 that there exist positive constants \( T > 0, M_i \) and \( m_i \) (defined by (3.2) and (3.6), respectively), such that for all \( t \geq T \),
\[ m_i < x_i^*(t) \leq M_i, \quad i = 1, 2, \quad m_3 < y^*(t) \leq M_3, \]
\[ m_i < y_i(t) \leq M_i, \quad i = 1, 2, 3. \]

We define
\[ V_1(t) = |\ln x_1^*(t) - \ln y_1(t)| + |\ln x_2^*(t) - \ln y_2(t)|. \]

Calculating the upper-right derivative of \( V_1(t) \) along the solution of (1.1), it follows for \( t \geq T \) that
\[ D^+ V_1(t) = \sum_{i=1}^{2} \left( \frac{\dot{x}_i^*(t)}{x_i^*(t)} - \frac{\dot{y}_i(t)}{y_i(t)} \right) \text{sgn} (x_i^*(t) - y_i(t)) \]
\[ = \text{sgn} (x_1^*(t) - y_1(t)) \left\{ - a_{11}(t)(x_1^*(t) - y_1(t)) - a_{13}(t)(y^*(t) - y_3(t)) \right. \]
\[ + D_1(t) \left( \frac{x_1^*(t)}{x_1^*(t)} - \frac{y_1(t)}{y_1(t)} \right) \left\} + \text{sgn} (x_2^*(t) - y_2(t)) \times \left\{ - a_{22}(t)(x_2^*(t) - y_2(t)) + D_2(t) \left( \frac{x_2^*(t)}{x_2^*(t)} - \frac{y_1(t)}{y_2(t)} \right) \right\} \]
\[ \leq - a_{11}(t)|x_1^*(t) - y_1(t)| + a_{13}(t)|y^*(t) - y_3(t)| - a_{22}(t)|x_2^*(t) - y_2(t)| \]
\[ - y_2(t)| + \tilde{D}_1(t) + \tilde{D}_2(t), \quad (4.4) \]
where

\[
\tilde{D}_1(t) = \begin{cases} 
D_1(t) \left( \frac{x_1^*(t)}{y_1(t)} - \frac{y_2(t)}{y_1(t)} \right), & x_1^*(t) > y_1(t), \\
D_1(t) \left( \frac{y_2(t)}{y_1(t)} - \frac{x_2^*(t)}{x_1^*(t)} \right), & x_1^*(t) < y_1(t),
\end{cases}
\]

\[
\tilde{D}_2(t) = \begin{cases} 
D_2(t) \left( \frac{x_1^*(t)}{x_2^*(t)} - \frac{y_1(t)}{y_2(t)} \right), & x_2^*(t) > y_2(t), \\
D_2(t) \left( \frac{y_1(t)}{y_2(t)} - \frac{x_1^*(t)}{x_2^*(t)} \right), & x_2^*(t) < y_2(t).
\end{cases}
\]

We estimate \(\tilde{D}_1(t)\) under the following three cases:

(i) If \(x_1^*(t) > y_1(t)\), then

\[
\tilde{D}_1(t) \leq \frac{D_1(t)}{x_1^*(t)} (x_2^*(t) - y_2(t)) \leq \frac{D_1^M}{m_1} |x_2^*(t) - y_2(t)|.
\]

(ii) If \(x_1^*(t) < y_1(t)\) then

\[
\tilde{D}_1(t) \leq \frac{D_1(t)}{y_1(t)} (y_2(t) - x_2^*(t)) \leq \frac{D_1^M}{m_1} |x_2^*(t) - y_2(t)|.
\]

(iii) If \(x_1^*(t) = y_1(t)\), a similar argument shows that the same conclusion as (i) and (ii) holds.

Combining the conclusions in (i)–(iii), we obtain

\[
\tilde{D}_1(t) \leq \frac{D_1^M}{m_1} |x_2^*(t) - y_2(t)|. \quad \text{(4.5)}
\]

A similar argument in the discussion above shows that

\[
\tilde{D}_2(t) \leq \frac{D_2^M}{m_2} |x_1^*(t) - y_1(t)|. \quad \text{(4.6)}
\]

It follows from (4.4)–(4.6) that

\[
D^+ V_1(t) \leq -a_{11}(t)|x_1^*(t) - y_1(t)| - a_{22}(t)|x_2^*(t) - y_2(t)|
+ a_{13}(t)|y^*(t) - y_3(t)| + \frac{D_1^M}{m_1} |x_2^*(t) - y_2(t)| + \frac{D_2^M}{m_2} |x_1^*(t) - y_1(t)|. \quad \text{(4.7)}
\]

Define

\[
V_{21}(t) = |\ln y^*(t) - \ln y_3(t)|. \quad \text{(4.8)}
\]

Calculating the upper-right derivative of \(V_{21}(t)\) along the solution of (1.1), we derive for \(t \geq T\) that
\[D^+ V_{21}(t) = \left( \frac{y^*(t)}{y^*(t)} - \frac{y_3(t)}{y_3(t)} \right) \text{sgn} \left( y^*(t) - y_3(t) \right)\]

\[= \text{sgn} \left( y^*(t) - y_3(t) \right) \left\{ -a_{33}(t)(y^*(t) - y_3(t)) + a_{31}(t)(x_1^*(t - \tau_1) - y_1(t - \tau_1)) \right\}\]

\[= \text{sgn} \left( y^*(t) - y_3(t) \right) \left\{ -a_{33}(t)(y^*(t) - y_3(t)) + a_{31}(t)(x_1^*(t - \tau_1) - y_1(t - \tau_1)) - y_1(t - \tau_1) + a_{33}(t) \int_{t - \tau_2}^{t} (y^*(u) - y_3(u)) \, du \right\}. \quad (4.9)\]

On substituting (1.1) into (4.9), we obtain

\[D^+ V_{21}(t) = \text{sgn} \left( y^*(t) - y_3(t) \right) \left\{ -a_{33}(t)(y^*(t) - y_3(t)) + a_{31}(t)(x_1^*(t - \tau_1) - y_1(t - \tau_1)) + a_{31}(t) \int_{t - \tau_2}^{t} (y^*(u) - y_3(u)) \, du \right\}. \quad (4.10)\]

It follows from (4.2) and (4.10) that for \( t \geq T + \tau \)

\[D^+ V_{21}(t) \leq -a_{33}(t)|y^*(t) - y_3(t)| + a_{31}(t)|x_1^*(t - \tau_1) - y_1(t - \tau_1)| \]

\[+ a_{33} \int_{t - \tau_2}^{t} \left[ (r_3(u) + a_{31}(u)M_1 + a_{33}(u)M_3)|y^*(u) - y_3(u)| + a_{31}(u)M_3|x_1^*(u - \tau_1) - y_1(u - \tau_1)| + a_{33}(u)M_3|y^*(u - \tau_2) - y_3(u - \tau_2)| \right] \, du. \quad (4.11)\]

Define

\[V_{22}(t) = \int_{t - \tau_1}^{t} a_{31}(s + \tau_1)|x_1^*(s) - y_1(s)| \, ds + \int_{t}^{t + \tau_2} \int_{t - \tau_2}^{t} a_{33}(s) \times \left\{ (r_3(u) + a_{31}(u)M_1 + a_{33}(u)M_3)|y^*(u) - y_3(u)| + a_{31}(u)M_3|x_1^*(u - \tau_1) - y_1(u - \tau_1)| + a_{33}(u)M_3|y^*(u - \tau_2) - y_3(u - \tau_2)| \right\} \, du \, ds. \quad (4.12)\]
We now define a Lyapunov functional

\[ D^+ V_2(t) \leq -a_{33}(t) |y^*(t) - y_3(t)| + a_{31}(t + \tau_1)|x_1^*(t) - y_1(t)| \]

\[ + \int_{t-t_2}^{t} a_{33}(s) \, ds \left\{ (r_3(t) + a_{31}(t)M_1 + a_{33}(t)M_3)|y^*(t) - y_3(t)| \right. \]

\[ + a_{31}(t + \tau_1)M_3 \int_{t-t_1}^{t-t_2} a_{33}(s) \, ds |x_1^*(t) - y_1(t)| + a_{33}(t + \tau_2)M_3 \]

\[ \left. \times \int_{t-t_2}^{t+\tau_2} a_{33}(s) \, ds |y^*(t) - y_3(t)| \right\}. \]  

(4.13)

We now define

\[ V_2(t) = V_1(t) + V_2(t) + V_3(t), \]  

(4.14)

in which

\[ V_3(t) = M_3 \int_{t-t_2}^{t} \int_{t-t_2}^{t+\tau_2} a_{33}(s) a_{33}(l + \tau_1)|x_1^*(l) - y_1(l)| \, ds \, dl + M_3 \]

\[ \times \int_{t-t_2}^{t} \int_{t+\tau_1}^{t+\tau_2} a_{33}(s) a_{33}(l + \tau_2)|y^*(l) - y_3(l)| \, ds \, dl. \]  

(4.15)

It then follows from (4.13)–(4.15) that for \( t \geq T + \tau \)

\[ D^+ V_2(t) \leq -a_{33}(t) |y^*(t) - y_3(t)| + a_{31}(t + \tau_1)|x_1^*(t) - y_1(t)| \]

\[ + \int_{t-t_2}^{t} a_{33}(s) \, ds \left\{ (r_3(t) + a_{31}(t)M_1 + a_{33}(t)M_3)|y^*(t) - y_3(t)| \right. \]

\[ + a_{31}(t + \tau_1)M_3 \int_{t-t_1}^{t-t_2} a_{33}(s) \, ds |x_1^*(t) - y_1(t)| + a_{33}(t + \tau_2)M_3 \]

\[ \left. \times \int_{t-t_2}^{t+\tau_2} a_{33}(s) \, ds |y^*(t) - y_3(t)| \right\}. \]  

(4.16)

We now define a Lyapunov functional \( V(t) \) as

\[ V(t) = V_1(t) + V_2(t). \]  

(4.17)

Then it follows from (4.7), (4.16) and (4.17) that for \( t \geq T + \tau \)

\[ D^+ V(t) \leq -\sum_{i=1}^{2} A_i(t)|x_i^*(t) - y_i(t)| - A_3(t)|y^*(t) - y_3(t)|, \]  

(4.18)

where \( A_1(t), A_2(t) \) and \( A_3(t) \) are defined in (4.1).

By hypothesis (H5), there exist constants \( \alpha_i > 0 \) (\( i = 1, 2, 3 \)) and \( T^* \geq T + \tau \), such that

\[ A_i(t) \geq \alpha_i > 0 \quad \text{for} \quad t \geq T^*. \]  

(4.19)
Integrating both sides of (4.18) on interval $[T^*, t]$,

$$V(t) + \sum_{i=1}^{2} \int_{T^*}^{t} A_i(s)|x_i^*(s) - y_i(s)| \, ds + \int_{T^*}^{t} A_3(s)|y^*(s) - y_3(s)| \, ds \leq V(T^*).$$

(4.20)

It follows from (4.19) and (4.20) that

$$V(t) + \sum_{i=1}^{2} \alpha_i \int_{T^*}^{t} |x_i^*(s) - y_i(s)| \, ds + \alpha_3 \int_{T^*}^{t} |y^*(s) - y_3(s)| \, ds \leq V(T^*) \quad \text{for } t \geq T^*.$$  

(4.21)

Therefore, $V(t)$ is bounded on $[T^*, \infty)$ and also

$$\int_{T^*}^{\infty} |x_i^*(s) - y_i(s)| \, ds < \infty, \quad i = 1, 2,$$

$$\int_{T^*}^{\infty} |y^*(s) - y_3(s)| \, ds < \infty.$$  

(4.22)

By Theorem 3.1, $|x_i^*(t) - y_i(t)|$ $(i = 1, 2)$ and $|y^*(t) - y_3(t)|$ are bounded on $[T^*, \infty)$.

On the other hand, it is easy to see that $\dot{x}_i^*(t)$ $(i = 1, 2)$, $\dot{y}^*(t)$ and $\dot{y}_i(t)$ ($i = 1, 2, 3$) are bounded for $t \geq T^*$. Therefore, $|x_i^*(t) - y_i(t)|$ $(i = 1, 2)$ and $|y^*(t) - y_3(t)|$ are uniformly continuous on $[T^*, \infty)$. By Barbalat’s Lemma (Lemma 1.2.2 and 1.2.3 [27]), we can conclude that

$$\lim_{t \to \infty} |x_i^*(t) - y_i(t)| = 0, \quad i = 1, 2;$$

$$\lim_{t \to \infty} |y^*(t) - y_3(t)| = 0.$$  

(4.23)

This completes the proof. $\square$

Finally, we give two examples to illustrate the feasibility of our main results.

**Example 1.** We consider the following Lotka–Volterra predator–prey model with prey dispersal and time delays

$$\begin{cases}
\dot{x}_1(t) = x_1(t)[5 - \sin t - 8x_1(t) - 5y(t)] + (0.1 - 0.01 \sin t)(x_2(t) - x_1(t)), \\
\dot{x}_2(t) = x_2(t)[6 - \cos t - 8x_2(t)] + (0.1 - 0.01 \cos t)(x_1(t) - x_2(t)), \\
\dot{y}(t) = y(t) \left[-0.1 + 0.01 \sin t + 4x_1 \left(t - \frac{1}{100}\right) - 8y \left(t - \frac{1}{100}\right)\right].
\end{cases}$$  

(4.24)

It is easy to examine that the coefficients in system (4.24) satisfy all assumptions in Theorems 2.1, 3.1, 4.1. Thus, by Theorem 4.1, system (4.24) has a unique
positive $2\pi$-periodic solution which is globally asymptotically stable. Using Shampine and Thompson’s program \texttt{dde23} in solving delay differential equations (see [39]), numerical simulation shows that system (4.24) has a unique $2\pi$-periodic solution which is globally asymptotically stable (see, Fig. 1).

**Example 2.** We consider another delayed periodic Lotka–Volterra diffusive predator–prey system

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t)[2 - \sin t - 3x_1(t) - 6y(t)] + (3 - \sin t)(x_2(t) - x_1(t)), \\
\dot{x}_2(t) &= x_2(t)[3 - \cos t - 2x_2(t)] + (4 - \cos t)(x_1(t) - x_2(t)), \\
\dot{y}(t) &= y(t) \left[-1 + 0.1 \sin t + 4x_1 \left( t - \frac{1}{100} \right) - 2y \left( t - \frac{1}{100} \right) \right].
\end{align*}
\]

(4.25)

Obviously, in system (4.25), $r_1(t) = 2 - \sin t < D_1(t) = 3 - \sin t$, $r_2(t) = 3 - \cos t < D_2(t) = 4 - \cos t$, this means that system (4.25) does not satisfy
(H2) and (H3). However, numerical simulation shows that system (4.25) still has at least one positive $2\pi$-periodic solution (see, Fig. 2).

**Remark.** In this paper, we have combined the effects of periodicity of the environment, prey dispersal and time delays on the dynamics of Lotka–Volterra predator–prey system. By using Gaines and Mawhin’s continuation theorem of coincidence degree theory and by means of a suitable Lyapunov functional, we have discussed the existence, uniqueness and global asymptotic stability of periodic solutions for a delayed periodic Lotka–Volterra predator–prey system with dispersion. By Theorem 2.1, we see that system (1.1) with initial conditions (1.2) will have at least one positive periodic solution if the intrinsic growth rate of the prey species in both patches and the conversion rate of the predator are high and the death rate of the predator and the dispersal rates of the prey are low. By Theorem 3.1, we see that the dispersion rates of the prey species have no effect on the uniform persistence of the system. By Theorem 4.1, we have shown that if system (1.1) admits positive $\omega$-periodic solutions, then it is unique and globally asymptotically stable provided that the density-
dependent coefficients of the prey and the predator are sufficiently large and the interspecific interaction coefficients, the dispersion rates of the prey and the time delays are sufficiently small.

We would like to mention here that Example 2 shows that our results in Theorems 2.1, 3.1, 4.1 have room for improvement. The next step in the study on predator–prey model with dispersion would be to incorporate time delays to the self-regulated terms of the prey in both patches. We leave these for our future work.

References

[34] R.M. May, Time delay versus stability in population models with two and three trophic levels, Ecology 4 (1973) 315–325.